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The moment generating function related to Bell polynomials

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Abstract: In this paper, an investigation has been conducted on the moment functions of various probability variables concerning the recently studied exponential λ-analogue. Using degenerate polynomials, generalized expressions for the mean and variance were derived and directly computed. Additionally, by leveraging Stirling numbers and Bell polynomials, connections with Poisson random variables were explored. Furthermore, meaningful results were obtained using covariance analysis. Moreover, we utilized these results to define a new form of Bell polynomials. We anticipate that investigating the properties of these polynomials in future research will yield new and valuable results.

Key words: Probability mass function, Degnerate exponential, Binomial random variable, Poisson random variavble, expectation, variance, moment generating function, Stirling numbers, Bell polynomials, covariance,

1 Introduction

For any $0 \neq \lambda \in \mathbf{N}$, it is well known that degenerate exponential is defined by

$$e_\lambda^x(t) = (1+\lambda t)^{\frac{x}{\lambda}} \tag{1}$$

$$= \sum_{n=0}^{\infty} \frac{(x)_{n,\lambda} t^n}{n!} \quad \text{(See[1])} \tag{2}$$

Here $(x)_{0,\lambda} = 1, (x)_{n,\lambda} = x(x-\lambda)\cdots(x-(n-1)\lambda)\ (n \geq 1)$.
And we can easily show that $\lim_{\lambda\to 0} e_\lambda^x(t) = e^{xt}, e_\lambda^1(t) = e_\lambda(t)$.
Let's consider its derivative with respect to t. Then we have

$$\begin{aligned}\frac{d}{dt}e_\lambda^x(t) &= \frac{d}{dt}(1+\lambda t)^{\frac{x}{\lambda}} \\ &= \frac{x}{\lambda}(1+\lambda t)^{\frac{x}{\lambda}-1}\cdot\lambda \\ &= x(1+\lambda t)^{\frac{x}{\lambda}-1} \\ &= \frac{x}{1+\lambda t}(1+\lambda t)^{\frac{x}{\lambda}} \\ &= \frac{x}{1+\lambda t}e_\lambda^x(t). \end{aligned} \tag{3}$$

Let X be the any random variable. The moment generating function $\Phi(t)$ for the random variable X is defined as follows, the n-th moment of X is given by $E[X^n]$

$$\begin{aligned}\sum_{n=0}^{\infty} E[X^n]\frac{t^n}{n!} &= E\left[\sum_{n=0}^{\infty} X^n \frac{t^n}{n!}\right] \\ &= E[e^{Xt}] : \text{generating function of moment}\end{aligned} \tag{4}$$

and

$$\begin{aligned}\Phi(t) &= \text{the moment generating function of random variable } X.\\ &= E[e^{Xt}]\\ &= \begin{cases}\sum_x e^{xt}p(x) & \text{if } X \text{ is discrete random variable,}\\ \int_{-\infty}^{\infty} e^{xt}f(x)dx & \text{if } X \text{ is continuous random variable.}\end{cases}\end{aligned} \tag{5}$$

Here $p(x)$ is the probability mass function of discrete random variable of X,
$f(x)$ is the probability density function of continuous random variable of X.
Note that

$$\begin{aligned}\Phi'(t) &= \frac{d}{dt}E[e^{Xt}]\\ &= E[\frac{d}{dt}e^{Xt}]\\ &= E[Xe^{Xt}].\end{aligned}$$

In particular

$$\Phi'(0) = E[X]. \tag{6}$$

From

$$\begin{aligned}\Phi''(t) &= \frac{d}{dt}E[Xe^{Xt}]\\ &= E[\frac{d}{dt}Xe^{Xt}]\\ &= E[X^2e^{Xt}].\end{aligned}$$

we get

$$\Phi''(0) = E[X^2]. \tag{7}$$

By continuing above process we can show that $\Phi^{(n)}(0) = E[X^n]$. In addiction, by (6), (7) and definition of variance we get following equation

$$\begin{aligned}Var(X) &= E[X^2] - (E[X])^2\\ &= \Phi''(0) - (\Phi'(0))^2.\end{aligned} \tag{8}$$

Suppose that n independent trials, each of which results in a "success" with probability p and in a "failure" with probability $1-p$, are to be performed. We call X is the binomial random variable with parameter n, p, which denoted by $X \backsim B(n,p)$. Then the probability mass function of X is given by

$$P[X=i] = p(i) = \binom{n}{i}p^i(1-p)^{n-i}, \quad (i = 0, 1, 2, \cdots, n) \quad \text{(See[2])}. \tag{9}$$

The moment generating function of binomial random variable X is given by

$$\begin{aligned}\Phi(t) &= E[e^{Xt}]\\ &= \sum_{k=0}^{n} e^{kt}p(k)\\ &= \sum_{k=0}^{n}\binom{n}{k}p^k(1-p)^{n-k}e^{kt}\\ &= (pe^t + 1 - p)^n.\end{aligned} \tag{10}$$

Hence, if $X \backsim B(n,p)$, then

$$\Phi(t) = (pe^t + 1 - p)^n. \tag{11}$$

From (11), we get the following results:

$\Phi'(t) = n(pe^t + 1 - p)^{n-1}pe^t,$
$\Phi''(t) = n(n-1)(pe^t + 1 - p)^{n-2}(pe^t)^2 + n(pe^t + 1 - p)^{n-1}pe^t$
Therefore, $\Phi'(0) = np$, $\Phi''(0) = n(n-1)p^2 + np$, and by (8)

$$\begin{aligned} Var(X) &= (n(n-1)p^2 + np) - (np)^2 \\ &= np(1-p). \end{aligned} \tag{12}$$

The Bell polynomials $Bel_n(x) = \sum_{k=0}^{n} S_2(n,k)x^n$ ($S_2(n,k)$: Stirling numbers of the second kind)(See[1]) are natural extensions of the Bell numbers which are a number of ways to partition a set with n elements into nonempty subsets. It is well known that the generating function of the Bell polynomials is given by

$$e^{x(e^t-1)} = \sum_{n=0}^{\infty} Bel_n(x)\frac{t^n}{n!} \quad \text{(See[2, 3, 4, 1])} \tag{13}$$

Definition of Poisson random variable is expresses the probability of a given number of events occurring in a fixed interval of time or space. X taking on one of the values $0, 1, 2, \cdots$. We denote $X \backsim Poi(\alpha)$ $(\alpha > 0)$. The probability mass function is

$$p(i) = e^{-\alpha}\frac{\alpha^i}{i!} \quad \text{(See[2, 3])}. \tag{14}$$

If we apply the process we used for the moment generating function of the binomial random variable to the Poisson random variable, we'll obtain the following result.

$$\begin{aligned} \Phi(t) &= E[e^{Xt}] \\ &= \sum_{k=0}^{\infty} e^{kt}p(k) \\ &= \sum_{k=0}^{\infty} e^{kt}\frac{\alpha^k e^{-\alpha}}{k!} \\ &= e^{-\alpha}\sum_{k=0}^{\infty}\frac{(\alpha e^t)^k}{k!} \\ &= e^{\alpha(e^t-1)} \end{aligned} \tag{15}$$

Note that if $X \backsim Poi(\alpha)$, then by (4), (13), (15)

$$E[X^n] = Bel_n(\alpha). \tag{16}$$

and

$$\Phi'(0) = E[X] = \alpha \quad \Phi''(0) = E[X^2] = \alpha^2 + \alpha. \tag{17}$$

Hence, by (8)

$$\begin{aligned} Var(X) &= E[X^2] - (E[x])^2 \\ &= \alpha^2 + \alpha - \alpha^2 \\ &= \alpha \end{aligned} \tag{18}$$

2 The λ-analogue moment generating function

We replace e^{Xt} by the $e_\lambda^X(t)$ in (4). Note that

$$E[g(x)] == \begin{cases} \sum_x g(x)p(x) & \text{if } X \text{ is discrete random variable,} \\ \int_{-\infty}^{\infty} g(x)f(x)dx & \text{if } X \text{ is continuous random variable.} \end{cases}$$

Now let

$$\Phi_\lambda(t) = E[e_\lambda^X(t)] \text{ be degenerate moment generating function of random variable } X. \tag{19}$$

Then

$$\begin{aligned} \Phi'_\lambda(t) &= \frac{d}{dt}\Phi_\lambda(t) \\ &= \frac{d}{dt}E[e_\lambda^X(t)] \\ &= E[\frac{d}{dt}e_\lambda^X(t)] \\ &= E[\frac{X}{1+\lambda t}e_\lambda^X(t)] \end{aligned} \tag{20}$$

and we obtain the following result.

Theorem 2.1 $\Phi'_\lambda(0) = E[X]$.

Similarly

$$\begin{aligned} \Phi''_\lambda(t) &= \frac{d}{dt}\Phi'_\lambda(t) \\ &= \frac{d}{dt}E[\frac{X}{1+\lambda t}e_\lambda^X(t)] \\ &= E[\frac{d}{dt}(\frac{X}{1+\lambda t}e_\lambda^X(t))] \\ &= E[-\frac{X\lambda}{(1+\lambda t)^2}e_\lambda^X(t) + (\frac{X}{1+\lambda t})^2 e_\lambda^X(t)] \end{aligned} \tag{21}$$

Putting $t = 0$ in (21), we get

Theorem 2.2 $\Phi''_\lambda(0) = E[-X\lambda + X^2] = E[X^2] - \lambda E[X]$.

By the definition of variance, we get

Theorem 2.3

$$\begin{aligned} Var(X) &= E[X^2] - (E[X])^2 \\ &= \Phi''_\lambda(0) + \lambda\Phi'_\lambda(0) - (\Phi'_\lambda(0))^2 \\ &= \Phi''_\lambda(0) + \Phi'_\lambda(0)(\lambda - \Phi'_\lambda(0)). \end{aligned} \tag{22}$$

Using the above theorems, we determine the expectation and variance of the random variable.

3 Binomial random variables

Let $X \backsim B(n,p)$, then by (9) we have.

$$\begin{aligned}\Phi_\lambda(t) &= E[e_\lambda^X(t)]\\ &= \sum_{k=0}^{n} e_\lambda^k(t)\binom{n}{k}p^k(1-p)^{n-k}\\ &= \sum_{k=0}^{n}\binom{n}{k}(pe_\lambda(t))^k(1-p)^{n-k}\\ &= (pe_\lambda(t)+1-p)^n. \end{aligned} \tag{23}$$

By (23),

$$\Phi'_\lambda(t) = n(pe_\lambda(t)+1-p)^{n-1}\frac{p}{1+\lambda t}e_\lambda(t). \tag{24}$$

and

$$\begin{aligned}\Phi''_\lambda(t) = n(n-1)(pe_\lambda(t)+1-p)^{n-2}(\frac{p}{1+\lambda t}e_\lambda(t))^2 - n(pe_\lambda(t)+1-p)^{n-1}\frac{\lambda p}{(1+\lambda t)^2}e_\lambda(t)\\ +n(pe_\lambda(t)+1-p)^{n-1}\frac{p}{(1+\lambda t)^2}e_\lambda(t). \end{aligned} \tag{25}$$

Thus, we obtain

Theorem 3.1

$$\begin{aligned}E[X] &= \Phi'_\lambda(0)\\ &= np, \end{aligned} \tag{26}$$

$$\begin{aligned}Var(X) &= \Phi''_\lambda(0)+\Phi'_\lambda(0)(\lambda-\Phi'_\lambda(0))\\ &= n(n-1)p^2 - n\lambda p + np + np(\lambda - np)\\ &= np(1-p). \end{aligned} \tag{27}$$

4 Poisson random variables

Let $X \backsim Poi(\alpha)$ $(\alpha > 0)$, then by (14) and expectation of degenerate exponential on X, we get

$$\begin{aligned}\Phi_\lambda(t) &= E[e_\lambda^X(t)]\\ &= \sum_{k=0}^{\infty} e_\lambda^k(t)\frac{e^{-\alpha}\alpha^k}{k!}\\ &= e^{-\alpha}\sum_{k=0}^{\infty}\frac{(e_\lambda(t)\alpha)^k}{k!}\\ &= e^{-\alpha}e^{e_\lambda(t)\alpha}\\ &= e^{\alpha(e_\lambda(t)-1)}. \end{aligned} \tag{28}$$

Similarly, by (29) we derive the following two equations:

$$\Phi'_\lambda(t) = e^{\alpha(e_\lambda(t)-1)}\frac{\alpha}{1+\lambda t}e_\lambda(t) \tag{29}$$

and

$$\Phi''_\lambda(t) = e^{\alpha(e_\lambda(t)-1)}((\frac{\alpha}{1+\lambda t}e_\lambda(t))^2 - \frac{\alpha\lambda}{(1+\lambda t)^2}e_\lambda(t) + \frac{\alpha}{(1+\lambda t)^2}e_\lambda(t)). \tag{30}$$

So we get

Theorem 4.1

$$\begin{aligned} E[X] &= \Phi'_\lambda(0), \\ &= \alpha \qquad (31) \\ Var(X) &= \Phi''_\lambda(0) + \Phi'_\lambda(0)(\lambda - \Phi'_\lambda(0)) \\ &= \alpha^2 - \alpha\lambda + \alpha + \alpha\lambda - \alpha^2 \\ &= \alpha. \qquad (32) \end{aligned}$$

Observe that

$$\begin{aligned} \Phi_\lambda(t) &= E[e_\lambda^X(t)] \\ &= E[\sum_{n=0}^{\infty} \frac{(X)_{n,\lambda}}{n!} t^n] \\ &= \sum_{n=0}^{\infty} E[(X)_{n,\lambda}] \frac{t^n}{n!} \\ &= e^{\alpha(e_\lambda(t)-1)}. \qquad (33) \end{aligned}$$

It is well known that the degenerate Bell polynomials are defined by

$$e^{x(e_\lambda(t)-1)} = \sum_{n=0}^{\infty} Bel_n(x|\lambda) \frac{t^n}{n!} \quad \text{(See[4])}. \qquad (34)$$

Then we get the following theorem by comparing coefficients on both sides on (33), (34)

Theorem 4.2

$$E[(X)_{n,\lambda}] = Bel_n(\alpha|\lambda), \quad \textit{where } X \backsim Poi(\alpha). \qquad (35)$$

The unsigned Stirling numbers of the first kind denoted by

$$\begin{bmatrix} n \\ k \end{bmatrix}$$

means the numbers of permutations on n-th elements with k-disjoint cycles And the Stirling numbers of the first kind are

$$S_1(n,k) = (-1)^{n-k} \begin{bmatrix} n \\ k \end{bmatrix}.$$

For the given non negative integers n, k with $n \geq k$, the λ-analogue of the Stirling numbers of the first kind are defined by

$$(x)_{n,\lambda} = \sum_{k=0}^{\infty} S_{1,\lambda}(n,k) x^k \quad \text{(See[5, 6])}. \qquad (36)$$

By (36), we get the following equation:

$$\frac{1}{k!}(\log(1+t))^k = \sum_{n=k}^{\infty} S_1(n,k) \frac{t^n}{n!} \quad \text{(See[6])}. \qquad (37)$$

We already know that

$$\log_\lambda(e_\lambda(t)) = e_\lambda(\log_\lambda(t)) = t \qquad (38)$$

$$\begin{aligned} \log_\lambda(1+t) &= \frac{1}{\lambda} \sum_{n=1}^{\infty} (\lambda)_n \frac{t^n}{n!} \\ &= \frac{1}{\lambda}((1+t)^\lambda - 1) \quad \text{(See[7])}. \qquad (39) \end{aligned}$$

Note that $e_\lambda(t)$ and $\log_\lambda(t)$ are inverse functions. From (38), (39) and (34), by replacing t to $\log_\lambda(1+t)$, we have

$$e^{\alpha(e_\lambda(\log_\lambda(1+t))-1))} = \sum_{k=0}^{\infty} Bel_k(\alpha|\lambda)\frac{(log_\lambda(1+t))^k}{k!} \tag{40}$$

$$= \sum_{k=0}^{\infty} Bel_k(\alpha|\lambda)\frac{(\frac{\log(1+t)}{\log(\lambda)})^k}{k!}$$

$$= \sum_{k=0}^{\infty} Bel_k(\alpha|\lambda)(\frac{1}{\log(\lambda)})^k \frac{1}{k!}(\log(1+t))^k$$

$$= \sum_{k=0}^{\infty} Bel_k(\alpha|\lambda)(\frac{1}{\log(\lambda)})^k \sum_{n=k}^{\infty} S_1(n,k)\frac{t^n}{n!} \quad \text{(by (38))}$$

$$= \sum_{n=0}^{\infty}(\sum_{k=0}^{n} Bel_k(\alpha|\lambda)(\frac{1}{\log(\lambda)})^k S_1(n,k))\frac{t^n}{n!}. \tag{41}$$

On the other hand.

$$e^{\alpha(e_\lambda(\log_\lambda(1+t))-1))} = e^{\alpha t}$$

$$= \sum_{n=0}^{\infty} \alpha^n \frac{t^n}{n!}. \tag{42}$$

By comparing coefficients on both sides at (41) and (42), we get the following theorem.

Theorem 4.3

$$\alpha^n = \sum_{k=0}^{n} Bel_k(\alpha|\lambda)(\frac{1}{\log(\lambda)})^k S_1(n,k), \quad where\ X \backsim Poi(\alpha). \tag{43}$$

5 Logarithmic random variables

In this section, we define a new random variable called the logarithm random variable with parameter $\alpha \in (0,1)$, the probability mass function of which is given by

$$p(i) = -\frac{1}{\log(1-\alpha)} \cdot \frac{\alpha^i}{i}, \quad (n = 1, 2, \cdots). \tag{44}$$

Applying above process to (44), we get

$$\Phi_\lambda(t) = E[e_\lambda^X(t)]$$

$$= \sum_{n=1}^{\infty}(-\frac{1}{\log(1-\alpha)} \cdot \frac{\alpha^n}{n})e_\lambda^n(t)$$

$$= -\frac{1}{\log(1-\alpha)}\sum_{n=1}^{\infty}(\alpha e_\lambda(t))^n \frac{1}{n}$$

$$= -\frac{1}{\log(1-\alpha)}\sum_{n=1}^{\infty}(-1)^n(-\alpha e_\lambda(t))^n \frac{1}{n}$$

$$= \frac{1}{\log(1-\alpha)}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{(-\alpha e_\lambda(t))^n}{n}$$

$$= \frac{1}{\log(1-\alpha)}\log(1-\alpha e_\lambda(t)). \tag{45}$$

Then

$$\Phi'_\lambda(t) = \frac{d}{dt}\Phi_\lambda(t)$$
$$= \frac{1}{\log(1-\alpha)} \cdot \frac{\alpha}{1-\alpha e_\lambda(t)} \cdot \frac{1}{1+\lambda t} e_\lambda(t), \tag{46}$$
$$\Phi''_\lambda(t) = \frac{d}{dt}\Phi'_\lambda(t)$$
$$= \frac{d}{dt}\left(\frac{1}{\log(1-\alpha)} \cdot \frac{\alpha}{1-\alpha e_\lambda(t)} \cdot \frac{1}{1+\lambda t} e_\lambda(t)\right)$$
$$= -\frac{\alpha}{\log(1-\alpha)}\left(\frac{\alpha}{(1-\alpha e_\lambda(t))^2} \cdot \left(\frac{e_\lambda(t)}{1+\lambda t}\right)^2 + \frac{1}{1-\alpha e_\lambda(t)} \cdot \frac{e_\lambda(t)(1-\lambda)}{(1+\lambda t)^2}\right). \tag{47}$$

Thus we have the following theorem

Theorem 5.1

$$E[X] = \Phi'_\lambda(0)$$
$$= -\frac{\alpha}{\log(1-\alpha)} \cdot \frac{1}{1-\alpha}, \tag{48}$$
$$Var(x) = \Phi''_\lambda(0) + \Phi'_\lambda(0)(\lambda - \Phi'_\lambda(0))$$
$$= -\frac{\alpha}{\log(1-\alpha)} \cdot \frac{1-\lambda+\alpha\lambda}{(1-\alpha)^2} - \frac{\alpha}{\log(1-\alpha)} \cdot \frac{1}{1-\alpha}\left(\lambda + \frac{\alpha}{\log(1-\alpha)} \cdot \frac{1}{1-\alpha}\right)$$
$$= -\frac{\alpha}{\log(1-\alpha)} \cdot \frac{1}{(1-\alpha)^2}\left(1 + \frac{\alpha}{\log(1-\alpha)}\right). \tag{49}$$
$$\tag{50}$$

where X is logarithm random variable with parameter $\alpha \in (0,1)$

6 λ-logarithmic random variables

For $\lambda \in (0,1)$, we note that

$$\log_\lambda(1-x) = \frac{1}{\lambda}\sum_{k=1}^{\infty}(\lambda)_k \frac{(-x)^k}{k!}$$
$$= \frac{1}{\lambda}\sum_{k=1}^{\infty}\lambda(\lambda-1)(\lambda-2)\cdots(\lambda-(k-1))(-1)^k \frac{x^k}{k!}$$
$$= \frac{1}{\lambda}\sum_{k=1}^{\infty}\lambda^k 1 \cdot \left(1-\frac{1}{\lambda}\right) \cdot \left(1-\frac{2}{\lambda}\right)\cdots\left(1-\frac{k-1}{\lambda}\right)(-1)^k \frac{x^k}{k!}$$
$$= \sum_{k=1}^{\infty}\frac{\lambda^{k-1}}{k!}(1)_{k,\frac{1}{\lambda}}(-1)^k x^k. \tag{51}$$

If X is a random variable with λ-logarithmic distribution with parameter $\alpha \in (0,1)$, then probability mass function of X is

$$P_\lambda[X=n] = -\frac{\lambda}{\log(1-\alpha\lambda)} \cdot \frac{\alpha^n \lambda^{n-1}}{n}. \tag{52}$$

By (51), (52) is the degenerate moment generating function of random variable X

$$\begin{aligned}\Phi_\lambda(t) &= E[e_\lambda^X(t)]\\ &= \sum_{k=1}^{\infty} e_\lambda^k(t)\left(-\frac{\lambda}{\log(1-\alpha\lambda)}\cdot\frac{\alpha^k\lambda^{k-1}}{k}\right)\\ &= -\frac{1}{\log(1-\alpha\lambda)}\sum_{k=1}^{\infty}\frac{(e_\lambda(t)\alpha\lambda)^k}{k}\\ &= \frac{1}{\log(1-\alpha\lambda)}\sum_{k=1}^{\infty}(-1)^{k-1}\frac{(-e_\lambda(t)\alpha\lambda)^k}{k}\\ &= \frac{1}{\log(1-\alpha\lambda)}\log(1-e_\lambda(t)\alpha\lambda). \end{aligned} \tag{53}$$

Then, we get

$$\begin{aligned}\Phi'_\lambda(t) &= \frac{d}{dt}\Phi_\lambda(t)\\ &= -\frac{1}{\log(1-\alpha\lambda)}\cdot\frac{1}{1-e_\lambda(t)\alpha\lambda}\cdot\frac{\alpha\lambda}{1+\lambda t}e_\lambda(t). \end{aligned} \tag{54}$$

and

$$\begin{aligned}\Phi''_\lambda(t) &= \frac{d}{dt}\Phi'_\lambda(t)\\ &= -\frac{1}{\log(1-\alpha\lambda)}\cdot\frac{1}{(1-e_\lambda(t)\alpha\lambda)^2}\left(\frac{\alpha\lambda}{1+\lambda t}e_\lambda(t)\right)^2-\frac{1}{\log(1-\alpha\lambda)}\cdot\frac{1}{1-e_\lambda(t)\alpha\lambda}\cdot\frac{\alpha\lambda}{(1+\lambda t)^2}e_\lambda(t). \end{aligned} \tag{55}$$

By (54), (55), we have

Theorem 6.1

$$\begin{aligned}E[X] &= \Phi'_\lambda(0)\\ &= \frac{1}{\log(1-\alpha\lambda)}\cdot\frac{1}{1-\alpha\lambda}(-\alpha\lambda), \end{aligned} \tag{56}$$

$$\begin{aligned}Var(x) &= \Phi''_\lambda(0)+\Phi'_\lambda(0)(\lambda-\Phi'_\lambda(0))\\ &= -\frac{1}{\log(1-\alpha\lambda)}\cdot\frac{\alpha\lambda}{1-\alpha\lambda}\left(\frac{1}{1-\alpha\lambda}+\lambda+\frac{1}{\log(1-\alpha\lambda)}\frac{\alpha\lambda}{1-\alpha\lambda}\right). \end{aligned} \tag{57}$$

7 Covariance of two random variables

Let X and Y be a random variables. Then the covariance of X and Y is defined by

$$\begin{aligned}Cov(X,Y) &= E[(X-E[X])(Y-E[Y])]\\ &= E[XY]-E[X]E[Y]. \end{aligned} \tag{58}$$

By (58), we obtain the following properties:

$$\begin{aligned}Cov(X,X) &= E[X^2]-E[X]E[X]\\ &= Var(X). \end{aligned} \tag{59}$$

and

$$\begin{aligned}Cov(X,Y) &= E[XY]-E[X]E[Y]\\ &= Cov(Y,X). \end{aligned} \tag{60}$$

If X and Y are independent then

$$\begin{aligned} Cov(X,Y) &= E[XY] - E[X]E[Y] \\ &= E[X]E[Y] - E[X]E[Y] \\ &= 0. \end{aligned}$$

We'll utilize the covariance calculating the variance of the sum of random variables. Let X_k be random variables for all $k \in \mathbb{N}$. Then

$$Var(\sum_{k=0}^{n} X_k) = \sum_{k=0}^{n} Var(X_k) + 2\sum_{i \nless j} Cov(X_i, X_j). \tag{61}$$

Consider the example: assume that $X \backsim N(0,1)$ is the normal random variable with parameter $0, 1$ and $Y = X^2$. Then

$$\begin{aligned} Cov(X,Y) &= E[XY] - E[X]E[Y] \\ &= E[X^3] - E[X]E[X^2]. \\ &= 0. \end{aligned} \tag{62}$$

So, in the above case we know X, Y are independent variables.

We apply above concept of Poisson random variable, to get some properties. Before that, we would like to examine a simple recurrence relation for the Bell polynomials. At (13), the generating function of the Bell polynomials are given by $e^{x(e^t-1)} = \sum_{n=0}^{\infty} Bel_n(x)\frac{t^n}{n!}$. Let us consider

$$\begin{aligned} e^{x(e^t-1)} - 1 &= \sum_{n=0}^{\infty} Bel_n(x)\frac{t^n}{n!} - 1 \\ &= \sum_{n=1}^{\infty} Bel_n(x)\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} Bel_{n+1}(x)\frac{t^{n+1}}{(n+1)!}. \end{aligned} \tag{63}$$

On the other hand

$$\begin{aligned} e^{x(e^t-1)} - 1 &= e^{x(e^t-1)}(1 - e^{-x(e^t-1)}) \\ &= (\sum_{l=0}^{\infty} Bel_l(x)\frac{t^n}{n!})(1 - \sum_{k=0}^{\infty} Bel_k(-x)\frac{t^k}{k!}) \\ &= \sum_{n=0}^{\infty} Bel_n(x)\frac{t^n}{n!} - \sum_{n=0}^{\infty}(\sum_{k=0}^{n}\binom{n}{k} Bel_{n-k}(x)Bel_k(-x))\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}(Bel_n(x) - \sum_{k=0}^{n}\binom{n}{k} Bel_{n-k}(x)Bel_k(-x))\frac{t^n}{n!}. \end{aligned} \tag{64}$$

Comparing the coefficients on both sides in (63),(64), we get

Theorem 7.1

$$Bel_{n+1}(x)\frac{t}{n+1} = Bel_n(x) - \sum_{k=0}^{n}\binom{n}{k} Bel_{n-k}(x)Bel_k(-x). \tag{65}$$

Let $X \backsim Poi(\alpha)$,($\alpha > 0$), and $Y = X^{n-1}$. Consider $Var(X+Y)$. By (16), (17), (18), (58) and (61), we have

$$\begin{aligned} Var(X+Y) &= Var(X) + Var(Y) + 2Cov(X,Y) \\ &= Var(X) + Var(X^{n-1}) + 2Cov(X, X^{n-1}) \\ &= \alpha + (E[X^{2(n-1)}] - (E[X])^2) + 2(E[X^n] - \alpha E[X^{n-1}]) \\ &= \alpha + (Bel_{2n-2}(\alpha) - (Bel_{n-1}(\alpha))^2) + 2(Bel_n(\alpha) - \alpha Bel_{n-1}(\alpha)) \\ &= \alpha + Bel_{2n-2}(\alpha) + 2Bel_n(\alpha) - Bel_{n-1}(\alpha)(Bel_{n-1}(\alpha) + \alpha). \end{aligned} \tag{66}$$

Theorem 7.2

$$Var(X+Y) = \alpha + Bel_{2n-2}(\alpha) + 2Bel_n(\alpha) - Bel_{n-1}(\alpha)(Bel_{n-1}(\alpha) + \alpha).$$

By utilizing the relationship in Theorem 7.1, we can adjust the index of the Bell polynomials in the equation at the Theorem 7.2 as desired.

Let us consider $X \backsim Poi(\alpha)$, $(\alpha > 0)$, $Y = (X-\lambda)_{n-1,\lambda}$. Then by (34),(58)

$$\begin{aligned} Cov(X,Y) &= E[XY] - E[X]E[Y] \\ &= E[(X)_{n,\lambda}] - \alpha E[(X-\lambda)_{n-1,\lambda}] \\ &= Bel_n(\alpha|\lambda) - \alpha E[(X-\lambda)_{n-1,\lambda}]. \end{aligned} \tag{67}$$

We note that

$$\begin{aligned} \sum_{n=0}^{\infty} E[(X-\lambda)_{n-1,\lambda}]\frac{t^{n-1}}{(n-1)!} &= E[\sum_{n=0}^{\infty}(X-\lambda)_{n-1,\lambda}\frac{t^{n-1}}{(n-1)!}] \\ &= E[\sum_{n=1}^{\infty}(X-\lambda)_{n-1,\lambda}\frac{t^{n-1}}{(n-1)!}] \\ &= E[\sum_{n=0}^{\infty}(X-\lambda)_{n,\lambda}\frac{t^{n}}{n!}] \\ &= E[e_{\lambda}^{X-\lambda}(t)] \\ &= \sum_{k=0}^{\infty} e_{\lambda}^{k-\lambda}(t)\frac{e^{-\alpha}\alpha^k}{k!} \\ &= e^{-\alpha}\sum_{k=0}^{\infty}\frac{(\alpha e_{\lambda}^{1-\frac{\lambda}{k}}(t))^k}{k!} \\ &= e^{-\alpha}e^{\alpha e_{\lambda}^{1-\frac{\lambda}{k}}(t)} \\ &= e^{\alpha(e_{\lambda}^{1-\frac{\lambda}{k}}(t))-1}. \end{aligned} \tag{68}$$

We define new type Bell polynomials by generating function as

$$e^{\alpha(e_{\lambda}^{1-\frac{\lambda}{k}}(t))-1} = \sum_{n=0}^{\infty} Bel_{n,1-\frac{\lambda}{k}}(\alpha|\lambda)\frac{t^n}{n!}. \tag{69}$$

By (68), (69), we get the following theorem.

Theorem 7.3

$$E[(X-\lambda)_{n-1,\lambda}] = Bel_{n,1-\frac{\lambda}{k}}(\alpha|\lambda), \quad where\ X \backsim Poi(\alpha). \tag{70}$$

If $\lim_{\lambda\to 0} E[(X-\lambda)_{n-1,\lambda}] = E[X^{n-1}]$., then $\lim_{\lambda\to 0} Bel_{n,1-\frac{\lambda}{k}}(\alpha|\lambda) = Bel_{n-1}(\alpha)$.

Substituting the above result into the equation (67) gives,

$$Cov(X,Y) = Bel_n(\alpha|\lambda) - \alpha Bel_{n,1-\frac{\lambda}{k}}(\alpha|\lambda). \tag{71}$$

By Theorem 4,2 and Theorem 7.2, we obtain

$$\begin{aligned} Var(X+(X-\lambda)_{n-1,\lambda}) &= Var(X)+Var((X-\lambda)_{n-1,\lambda})+2Cov(X,(X-\lambda)_{n-1,\lambda}) \\ &= \alpha+Var((X-\lambda)_{n-1,\lambda})+2(Bel_n(\alpha|\lambda)-\alpha Bel_{n,1-\frac{\lambda}{k}}(\alpha|\lambda)) \\ &= \alpha+(E[((X-\lambda)_{n-1,\lambda})^2]-(E[(X-\lambda)_{n-1,\lambda}])^2)+2(Bel_n(\alpha|\lambda)-\alpha Bel_{n,1-\frac{\lambda}{k}}(\alpha|\lambda)) \\ &= \alpha+(E[((X-\lambda)_{n-1,\lambda})^2]-(Bel_{n,1-\frac{\lambda}{k}}(\alpha|\lambda)))^2)+2(Bel_n(\alpha|\lambda)-\alpha Bel_{n,1-\frac{\lambda}{k}}(\alpha|\lambda)). \end{aligned} \tag{72}$$

Thus, we arrive at the following result.

Theorem 7.4

$$Var(X+(X-\lambda)_{n-1,\lambda}) == \alpha+(E[((X-\lambda)_{n-1,\lambda})^2]-(Bel_{n,1-\frac{\lambda}{k}}(\alpha|\lambda)))^2)+2(Bel_n(\alpha|\lambda)-\alpha Bel_{n,1-\frac{\lambda}{k}}(\alpha|\lambda)).$$

Now, by inputting the necessary foundational values into the expressions we obtained, we can derive results for the variance.

8 Conclusion

Through the above discussions, we generalized the expressions for the expectations and variances of various random variables using degenerate formulas, exploring the outcomes. By employing moment functions, we investigated the relations between probability variables and well-known Stirling numbers and Bell polynomials. As a result, we observed that as λ goes to 0 in the limit, the expectations and variances of conventional random variables converge to the same values. We expressed the covariance of Poisson random variables using Bell polynomials, enabling us to obtain calculated results tailored to real-world scenarios by inserting desired values for each n, λ, and α. In future research, we anticipate obtaining meaningful results by investigating the properties and specific values of the newly defined Bell polynomial from this paper.

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